GLOBAL EXISTENCE AND EXPONENTIAL DECAY RESULTS OF A VISCOELASTIC HEAT EQUATION WITH LOGARITHMIC SOURCES

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ABSTRACT

In this paper, we study a nonlinear viscoelastic heat equation with logarithmic sources. By introducing a family of potential wells, we prove the global existence and exponential decay for solutions with initial data in the potential wells.

Keywords: Nonlinear viscoelastic heat equation, global existence, exponential decay, logarithmic sources.

1. INTRODUCTION

In this paper, we study the following heat equation with viscoelastic term and logarithmic nonlinearity

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = ku\ln|u| - u, & \text{in} \quad \Omega\times(0,T), \\ u = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $0 < T < \infty$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and k is a postive real number. The kernel g satisfies some conditions will be specified later.

The first equation in (1.1) without viscoelastic term (that is, the relaxation function g vanishes) has the form

$$u_t - \Delta u = f(u), \tag{1.2}$$

where $f(u) = ku\ln|u| - u.$ Related to these type equations with logarithmic nonlinearity source $f(u) = u\ln|u|,$ we refer the readers to [1] and references therein. In [1], by using the logarithmic Sobolev inequality and a family of potential wells, Chen at al obtained the global existence, decay estimate and blow-up at $+\infty$ of solutions under some suitable conditions.

In the case with presence of the memory term $\int_0^t g(t-s)\Delta u(s)ds,$ the equation in (1.1) has the form

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = f(u), \tag{1.3}$$

where $f(u)=ku\ln|u|-u.$ Concerning the results of global existence and blow-up in finite time or decay property for the solutions of problems related to equation (1.3) with the source term is the power functions, i.e, $f(u)=|u|^{p-2}u,$ or power-like functions satisfying:

(1) $f\in C^1$ and $f(0)=f'(0)=0.$

(2) (a) f is monotone and is convex for $u>0,$ and concave for $u<0;$ or (b) f is convex.

(3) $(p+1)\int_0^u f(z)dz\le uf(u),$ and $|uf(u)|\le\kappa\int_0^u f(z)dz,$ where

$$2<p+1\le\kappa<2^*=:\begin{cases}\infty, & \text{if } n\le 2,\\ \dfrac{2n}{n-2}, & \text{if } n\ge 3,\end{cases}$$

have attracted a great deal of attention in last several decades. For instance in [4], Messaoudi studied the equation (1.3) in the case $f(u)=b|u|^{p-2}u,$ associated with homogeneous Dirichlet boundary condition. If the relaxation function g is assumed to be nonnegative; $g'(t)\le 0$ and

$$\int_0^\infty g(s)ds<\frac{p-2}{p-3/2},$$

the author proved the blow-up of weak solution with positive initial energy by the convexity method.

In [8], Truong and Y considered the equation (1.3) with $f(u)$ in the general polynomial satisfied above conditions and they obtained the decay property and blow up in finite time for solutions. We refer to [2, 3, 6] for further results on this type of equations. However, when $f(u)$ is a logarithmic nonlinear function, i.e. $f(u)=ku\ln|u|-\alpha u,$ as far as we know, there is no results on this aspect of the global solution and decay property for the problem (1.1). The main difficulty in this case is that the method of potential wells in [4, 7] will be not suitable for the problem (1.1), since the nonlinear function $f(u)=ku\ln|u|-\alpha u$ doesn't satisfy the conditions (1) - (3).

Motivated by all these works, we consider the problem (1.1), by using the potential wells and the Sobolev logarithmic Sobolev inequality (see [1] Proposition 1.1), we obtain the global existence and by contructing a suitable Lyapunov functional to obtain the exponential decay property of solutions under starting in the stable set.

This paper is organized as follows. In the next section, we present some assumptions, notations and preparing results. In Section 3, we establish potential wells which is related to the logarithmic source of the problem (1.1). In Section 4, we obtain the global existence of the solutions and the last section, we prove the exponential decay of the solution.

2. PRELIMINARIES

Throughout this paper, we denote $L^p(\Omega)$-norm by $\|\cdot\|_p,$ especially $\|\cdot\|=\|\cdot\|_{L^2(\Omega)},$ and let $\langle\cdot,\cdot\rangle$ denote L^2-inner product.

First, for any $u_0 \in H_0^1(\Omega)$ and any $a > 0$, we have following logarithmic Sobolev inequality (see [1] Proposition 1.1):

$$2\int_\Omega |u(x)|^2 \ln\left(\frac{|u(x)|}{\|u\|_{L^2(\Omega)}}\right)dx + n(1+\ln a)\|u\|^2_{L^2(\Omega)} \le \frac{a^2}{\pi}\int_\Omega |\nabla u(x)|^2 dx. \qquad (2.1)$$

To state our main results, we need the following definitions.

Definition 2.1. $u = u(x,t)$ is called a weak solution of problem (1.1) on $\Omega\times[0,T)$, if $u \in L^\infty(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ and sastifies the problem (1.1) in distribution sense, i.e.

$$\langle u_t, v\rangle + \langle \nabla u, \nabla v\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle ds + \langle u, v\rangle = \langle ku\ln|u|, v\rangle, \qquad (2.2)$$
$$\forall v \in H_0^1(\Omega),\ t\in(0,T),$$

where $u(x,0) = u_0(x) \in H_0^1(\Omega)$.

Definition 2.2. Let $u(x,t)$ be a weak solution of problem (1.1). We define the maximal existence time T of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \le t < \infty$, then $T = +\infty$.

(ii) If there exists a $t_0 \in (0,\infty)$ such that $u(x,t)$ exists for $0 \le t < t_0$, but doesn't exists at $t = t_0$, then $T = t_0$.

The following conditions are the basis hypotheses to establish the main results of this paper.

(G) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying:

(i) $g(t) \ge 0,\ g'(t) \le 0,\ 1 - \int_0^\infty g(s)ds = \ell > 0;$

(ii) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t) \le -\xi(t)g(t),\ \xi'(t) \le 0,\ \int_0^\infty \xi(t)dt = +\infty,\ \forall t > 0.$$

(K) The constant k satisfies $0 < k < k_0$, where k_0 is the positive real number satisfying:

$$\sqrt{\frac{2\pi\ell}{k_0}} = e^{-1-\frac{2}{nk_0}}.$$

Remark 2.3. The function $f(s) = \sqrt{\frac{2\pi\ell}{s}} - e^{-1-\frac{2}{ns}}$ is a continuous and decreasing function on $(0,\infty)$, with

$$\lim_{s\to 0^+} f(s) = +\infty \text{ and } \lim_{s\to+\infty} f(s) = -e^{-1}.$$

Then, there exists a unique $k_0 > 0$ such that $f(k_0) = 0$. Moreover,

$$e^{-1-\frac{2}{ns}} < \sqrt{\frac{2\pi\ell}{s}},\ \forall s \in (0, k_0).$$

Next, we define functionals on $H_0^1(\Omega)$ as follows:

$$\begin{aligned} E(t) &= \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 \\ &\quad + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{2}\int_\Omega u^2 \ln|u|^k\, dx + \frac{k+2}{4}\|u(t)\|^2, \end{aligned} \tag{2.3}$$

$$J(u(t)) = \frac{1}{2}\left(\ell\|\nabla u(t)\|^2 - \int_\Omega u^2 \ln|u|^k\, dx + \|u(t)\|^2\right) + \frac{k}{4}\|u(t)\|^2, \tag{2.4}$$

$$I(u(t)) = \ell\|\nabla u(t)\|^2 - \int_\Omega u^2 \ln|u|^k\, dx + \|u(t)\|^2, \tag{2.5}$$

where $(g \circ v)(t) = \int_0^t g(t-s)\|v(t)-v(s)\|^2\, ds.$

Then it is obvious that

$$J(u(t)) = \frac{1}{2} I(u(t)) + \frac{k}{4}\|u(t)\|^2 \tag{2.6}$$

and

$$E(t) \geq \frac{1}{2} I(u(t)) + \frac{1}{2}(g \circ \nabla u)(t) + \frac{k}{4}\|u(t)\|^2. \tag{2.7}$$

Lemma 2.4. $E(t)$ is a nonincreasing function for $t \geq 0$ and

$$E'(t) = -\|u_t\|^2 - \frac{1}{2}g(t)\ \|\nabla u(t)\|^2 + \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \tag{2.8}$$

Proof. Multiply $(1.1)_1$ by u_t and integrating on $\Omega,$ we obtain (2.8) after some simple calculations. □

3. POTENTIAL WELLS

In this section, we establish the potentail wells which is related to the logarithmic nonlinear term $u\ln|u|.$

First, we define

$$\begin{aligned} \mathcal{N} &= \left\{u \in H_0^1(\Omega) : I(u) = 0,\ \|\nabla u\| \neq 0\right\}, \\ d &= \inf\left\{\sup_{\lambda \geq 0} J(\lambda u) : u \in H_0^1(\Omega),\ \|\nabla u\| \neq 0\right\}. \end{aligned} \tag{3.1}$$

Then we have the following lemma for well-defined of the potential well d.

Lemma 3.1. $0 < d = \inf_{u \in \mathcal{N}} J(u).$

Proof. Let $u \in H_0^1,\ \|\nabla u\| \neq 0,$ we have

$$J(\lambda u) = \frac{\lambda^2}{2}\ell\|\nabla u\|^2 - \frac{\lambda^2}{2}\int_\Omega u^2 \ln|u|^k\, dx - \frac{k\lambda^2}{2}(\ln\lambda)\|u\|^2 + \lambda^2\frac{k+2}{4}\|u\|^2, \text{ for all } \lambda \geq 0.$$

Then

$$\frac{d}{d\lambda}J(\lambda u)=\lambda\left[\ell\|\nabla u\|^2-\int_\Omega u^2\ln|u|^k\,dx+\|u\|^2-k\left(\ln\lambda\right)\|u\|^2\right],$$

$$\frac{d}{d\lambda}J(\lambda u)=0\Leftrightarrow\lambda=\lambda^*=\exp\left(\frac{I\left(u\right)}{k\|u\|^2}\right).$$

It is easy to show that

$$\sup_{\lambda\geq 0}J(\lambda u)=J(\lambda^* u).$$

On the other hand, it follows from (2.5) and (3.1), that $\lambda^* u\in\mathcal{N}$ and consequently

$$\sup_{\lambda\geq 0}J(\lambda u)=J(\lambda^* u)\geq\inf_{u\in\mathcal{N}}J(u).$$

This implies

$$\inf\left\{\sup_{\lambda\geq 0}J(\lambda u):u\in H_0^1,\ \|\nabla u\|\neq 0\right\}\geq\inf_{u\in\mathcal{N}}J(u).$$

In order to prove the inverse inequality, we note that, for each $u\in\mathcal{N},$ the function $\lambda\mapsto J(\lambda u)$ attains its maximum at $\lambda_0=1.$ So we obtain

$$\inf\left\{\sup_{\lambda\geq 0}J(\lambda u):u\in H_0^1,\ \|\nabla u\|\neq 0\right\}\leq\inf\left\{\sup_{\lambda\geq 0}J(\lambda u):u\in\mathcal{N}\right\}=\inf\{J(u):u\in\mathcal{N}\}$$

and we get $d=\inf_{u\in\mathcal{N}}J(u).$ The proof is complete. □

We now can define the modified stable set as in [7].

$$\mathcal{W}=\left\{u\in H_0^1\left(\Omega\right):I(u)>0\right\}\cup\{0\}.$$

Lemma 3.2. If $0<\|\nabla u\|\leq\vartheta,$ then $I(u)\geq 0,$ where $\vartheta=\lambda_1^{1/2}\left(\frac{2\pi\ell}{k}\right)^{\frac{n}{4}}e^{\frac{n}{2}}$ and λ_1 is the first eigenvalue of the following equation

$$\begin{cases}-\Delta u=\lambda u,\ \text{in }\Omega,\\ u=0,\qquad\text{on }\partial\Omega.\end{cases}\tag{3.2}$$

Proof. Using the Sobolev inequality (2.1), for any $a>0,$ we have

$$\begin{aligned}I(u)&=\ell\|\nabla u\|^2-\int_\Omega u^2\ln|u|^k\,dx+\|u\|^2\\&\geq\ell\|\nabla u\|^2-k\int_\Omega u^2\left(\ln\frac{|u|}{\|u\|}+\ln\|u\|\right)dx+\|u\|^2\\&\geq\left(\ell-k\frac{a^2}{2\pi}\right)\|\nabla u\|^2+\|u\|^2+\frac{kn(1+\ln a)}{2}\|u\|^2-k\|u\|^2\ln\|u\|.\end{aligned}\tag{3.3}$$

Since $1\geq\ell-k\frac{a^2}{2\pi},$ we can deduce from (3.3) that

$$I(u) \geq \left(\ell - k\frac{a^2}{2\pi}\right)\left(\|\nabla u\|^2 + \|u\|^2\right) + \frac{kn(1+\ln a)}{2}\|u\|^2 - k\|u\|^2 \ln\|u\|. \quad (3.4)$$

Taking $a = \sqrt{\frac{2\pi\ell}{k}}$ in (3.4), we gain

$$I(u) \geq k\left(\frac{n\left(1+\ln\sqrt{\frac{2\pi\ell}{k}}\right)}{2} - \ln\|u\|\right)\|u\|^2. \quad (3.5)$$

If $0 < \|\nabla u\| \leq \vartheta,$ then $\|u\| \leq \left(\frac{2\pi\ell}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}},$ so we have $I(u) \geq 0$ from (3.5). The proof is complete. □

Lemma 3.3. For any $u \in H_0^1, \|u\| \neq 0,$ and let $j(\lambda) = J(\lambda u).$ Then we have

$$I(\lambda u) = \lambda j'(\lambda)\begin{cases} > 0, \ 0 \leq \lambda < \lambda^*, \\ = 0, \ \lambda = \lambda^*, \\ < 0, \ \lambda^* < \lambda < +\infty. \end{cases}$$

Proof. We have

$$j(\lambda) = J(\lambda u) = \frac{\lambda^2}{2}\ell\|\nabla u\|^2 - \frac{\lambda^2}{2}\int_\Omega u^2 \ln|u|^k \, dx - \frac{k\lambda^2}{2}(\ln\lambda)\|u\|^2 + \frac{\lambda^2(k+2)}{4}\|u\|^2.$$

Since $\|u\| \neq 0,$ then $j(0) = 0,$ $j(+\infty) = -\infty,$ and

$$I(\lambda u) = \lambda\frac{d}{d\lambda}J(\lambda u) = \lambda j'(\lambda)$$
$$= \lambda^2\ell\|\nabla u(t)\|^2 - \lambda^2\int_\Omega u^2 \ln|u| \, dx + \lambda^2 \ \|u\|^2 - k\lambda^2 \ln\lambda\|u\|^2.$$

So, we have

$$I(\lambda u) = \lambda j'(\lambda)\begin{cases} > 0, \ 0 \leq \lambda < \lambda^*, \\ = 0, \ \lambda = \lambda^*, \\ < 0, \ \lambda^* < \lambda < +\infty. \end{cases} \quad □$$

Lemma 3.4. For the constant d in (3.1), we have

$$d \geq M = \frac{k}{4}\left(\frac{2\pi\ell}{k}\right)^{\frac{n}{2}} e^n.$$

Proof. We have

$$\sup_{\lambda \geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2}I(\lambda^* u) + \frac{k}{4}\|\lambda^* u\|^2. \quad (3.6)$$

From (3.5) and Lemma 3.3, we obtain

$$0 = I(\lambda^* u) \geq k\left(\frac{n\left(1+\ln\sqrt{\frac{2\pi\ell}{k}}\right)}{2} - \ln\left\|\lambda^* u\right\|\right)\left\|\lambda^* u\right\|^2,$$

then

$$\left\|\lambda^* u\right\| \geq \left(\frac{2\pi\ell}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}}. \tag{3.7}$$

From (3.1), (3.6) and (3.7), it implies that

$$d \geq M = \frac{k}{4}\left(\frac{2\pi\ell}{k}\right)^{\frac{n}{2}} e^n. \ \square$$

4. GLOBAL EXISTENCE

Proposition 4.1. Let u be a weak solution of problem (1.1) and $u_0 \in H_0^1(\Omega)$. Suppose that $E(0) < d$ and $u_0 \in \mathcal{W}$ then $u(t) \in \mathcal{W}$, for $0 \leq t \leq T$, where t is the maximum existence time of $u(t)$.

Proof. Let u be a weak solution problem of (1.1) under condition $E(0) < d$, $u_0 \in \mathcal{W}$ and T be the maximum existence time of $u(t)$. Then by Lemma 2.4 and (2.4), we have

$$J(u(t)) \leq E(t) \leq E(0) < d$$

We shall prove $I(u(t)) > 0$ for $0 < t < T$. Arguing by contradiction, suppose that there exists $t_0 \in (0,T)$ such that $I(u(t_0)) < 0$. By the continuity of $I(u(t))$ in t, there is a $t^* \in (0,T)$ to make $I(u(t^*)) = 0$.

However from the definition of d, one has

$$d > E(0) \geq E(u(t^*)) \geq J(u(t^*)) \geq d,$$

which is a contradition. $\square$

We now state the existence of solution to (1.1) which can be obtained by Faedo–Galerkin methods.

Theorem 4.2. If $u_0 \in H_0^1(\Omega)$, $E(0) < M$ $u_0 \in \mathcal{W}$, and (G, (i)), (K) hold. Then the problem (1.1) has a global weak solution $u \in L^\infty(0,+\infty; H_0^1(\Omega))$ with $u_t \in L^2(0,+\infty; L^2(\Omega))$.

Proof. Let $\{w_j\}$ be the Galerkin basis for $-\Delta$ in H_0^1. We find the approximate solution of the problem (1.1) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{4.1}$$

where the coefficients functions c_{mj}, $1 \leq j \leq m$, satisfy the system of integro-differential equations

$$\begin{cases} \langle u_{mt}, w_j\rangle + \left\langle \nabla u_m, \nabla w_j \right\rangle - \int_0^t g(t-s)\left\langle \nabla u_m(s), \nabla w_j \right\rangle ds \\ \qquad\qquad\qquad + \left\langle u_m, w_j \right\rangle = \left\langle k u_m \ln|u_m|, w_j \right\rangle, 1 \le j \le m, \\ u_m(0) = u_{0m}, \end{cases} \tag{4.2}$$

where

$$u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \text{ strongly in } H_0^1. \tag{4.3}$$

According to Schauder's fixed point theorem, we find that (4.1) lead to a system of integro-differential equations in the variable t that has a local solution $u_m(t)$ on $[0, T_m]$. First, by $u_0 \in \mathcal{W}$ and

$$\frac{1}{2}(1-\ell)\ \|\nabla u_{0m}\|^2 + J(u_{0m}) = E(u_{0m}),$$

thanking (4.3) we have $u_{0m} \in \mathcal{W}$ for sufficiently large m and $E(u_{0m}) < M$.

Multiplying the j^{th} equation of (4.2) by $c'_{mj}(t)$ and summing up with respect to j, for sufficiently large m, we obtain

$$\begin{aligned} \int_0^t \|u_{mt}(s)\|^2 ds + J(u_m(t)) &\le \int_0^t \|u_{mt}(s)\|^2 ds + E(u_m(t)) \\ &= E(u_{0m}) - \frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2 ds + \frac{1}{2}\int_0^t \left(g' \circ \nabla u_m\right)(s) ds \qquad (4.4) \\ &< M \le d,\ 0 \le t < +\infty. \end{aligned}$$

From (4.4) and Proposition 4.1, we have $u_m(t) \in \mathcal{W}$ for sufficiently large m and $0 \le t < \infty$. Combining (2.6) with (4.4), we derive

$$\|u_m(t)\|^2 < \frac{4}{k} M.$$

Now, using logarithmic Sobolev inequality (2.1), we obtain

$$E(t) \ge \frac{1}{2}\left(\ell - \frac{ka^2}{2\pi}\right)\|\nabla u(t)\|^2 + \frac{1}{2}\left(g \circ \nabla u\right)(t) + \frac{1}{4}\left[k + 2 + kn(1+\ln a) - k\ln\left(\frac{4}{k}M\right)\right]\|u(t)\|^2. \tag{4.6}$$

Choosing

$$\max\left\{ e^{-1-\frac{1}{n}-\frac{2}{nk}}, e^{-\frac{1}{n}-\frac{2}{nk}}\sqrt{\frac{2\pi\ell}{k}} \right\} < a < \sqrt{\frac{2\pi\ell}{k}} \tag{4.7}$$

will make

$$\ell - \frac{ka^2}{2\pi} > 0, \quad k + 2 + kn(1+\ln a) > 0 \quad \text{and} \quad k + 2 + kn(1+\ln a) - k\ln\left(\frac{4}{k}M\right) > 0.$$

Hence, we can deduce from (4.4) and (4.6) that

$$\|\nabla u_m\|^2 \le \frac{4\pi}{2\pi\ell - ka^2} E\left(u_m(t)\right) \le \frac{4\pi}{2\pi\ell - ka^2} M = C_M.$$

We also get from (4.4) that

$$\int_0^t \|u_{mt}(s)\|^2 \, ds < M. \qquad (4.9)$$

On the other hand, by direct calculation, we have

$$\begin{aligned}\int_\Omega \left(u_m \ln|u_m|\right)^2 dx &= \int\limits_{\{x\in\Omega:|u_m(x)|\le 1\}} \left(u_m \ln|u_m|\right)^2 dx + \int\limits_{\{x\in\Omega:|u_m(x)|>1\}} \left(u_m \ln|u_m|\right)^2 dx \\ &\le e^{-2}|\Omega| + \left(\frac{n-2}{2}\right)^2 S^{2^*} \|\nabla u_m\|^{2^*} \le C_M,\end{aligned} \qquad (4.10)$$

where S is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$.

By (4.8) – (4.10), we deduce that, there exists a subsequence of $\{u_m\}$, still denote by $\{u_m\}$ such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,\infty;H_0^1) \quad \text{weakly*}, \\ u_{mt} \to u_t & \text{in} \quad L^2(0,\infty;L^2) \quad \text{weakly}, \\ u_m \ln|u_m| \to u \ln|u| & \text{in} \quad L^\infty(0,\infty;L^2) \quad \text{weakly*}. \end{cases} \qquad (4.11)$$

Hence in $(4.2)_1$, for j fixed and $m \to \infty$, we have

$$\langle u_t, w_j\rangle + \left\langle \nabla u, \nabla w_j \right\rangle - \int_0^t g(t-s)\left\langle \nabla u(s), \nabla w_j \right\rangle ds + \langle u, w_j\rangle = \left\langle ku\ln|u|, w_j \right\rangle, \ \forall j = 1,2,... \qquad (4.12)$$

On the other hand, from (4.3), we obtain $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$. Then u is a global weak solution of the problem (1.1). □

5. EXPONENTIAL DECAY

We begin this section by the following lemma which is helpful to the proof of Theorem 5.4.

Lemma 5.1. Suppose that assumptions in Theorem 4.2 hold and $E(0) < d$. Then there exists a positive constant C, such that

$$I(u(t)) \ge \frac{k}{2}\ln\left(\frac{d}{E(0)}\right)\|u(t)\|^2 \quad \text{and} \quad E(t) \le C\ \|\nabla u(t)\|^2.$$

Proof. Since $u_0 \in \mathcal{W}$ and $E(0) < d$. By Theorem 4.2, we have that $u(t) \in \mathcal{W}$ for $t \ge 0$ and $I\left(u(t)\right) > 0$.

Put $\phi(\lambda) = k\ln\lambda\|u\|^2$ for $\lambda \in (0,\infty)$. Then for any $\lambda > 0$, we have

$$\begin{aligned} I(\lambda u) &= \lambda^2\left(\ell\|\nabla u\|^2 - \int_\Omega u^2 \ln|u|^k \, dx + \|u\|^2 - k\ln\lambda.\|u\|^2\right) \\ &= \lambda^2\left(I(u) - \phi(\lambda)\right). \end{aligned} \qquad (5.13)$$

Since $I(u) > 0$, we get

$$\phi(1) = 0 < I(u) = \ell \|\nabla u\|^2 - \int_\Omega u^2 \ln|u|^k \, dx + \|u\|^2. \tag{5.14}$$

Moreover, it is easy to see that ϕ is continuous, increasing and $\lim_{\lambda\to\infty} \phi(\lambda) = +\infty$, which combining with (5.14) to imply that there exist a $\lambda^* > 1$ such that $\phi(\lambda_*) = I(u)$ and $I\left(\lambda^* u\right) = 0.$ Taking into account this fact and the following estimate

$$0 = I(\lambda^* u(t)) = \left(\lambda^*\right)^2 I(u) - k\left(\lambda^*\right)^2 \ln \lambda^* . \|u(t)\|^2$$

we imply that

$$I\left(u(t)\right) = k \ln \lambda^* . \|u(t)\|^2. \tag{5.15}$$

To end the proof, it remains to estimate $\lambda^*.$ By variational characterization of $d,$ we have

$$\begin{aligned} d &\le J\left(\lambda^* u(t)\right) = \frac{1}{2} I\left(\lambda^* u(t)\right) + \frac{k}{4}\left(\lambda^*\right)^2 \|u(t)\|^2 \\ &\le \frac{k}{4}\left(\lambda^*\right)^2 \|u(t)\|^2. \end{aligned} \tag{5.16}$$

On the other hand, by the non-increasing property of functional energy $E(t),$ we have that

$$\begin{aligned} E(0) \ge E(t) \ge J\left(u(t)\right) &= \frac{1}{2} I\left(u(t)\right) + \frac{k}{4}\|u(t)\|^2 \\ &\ge \frac{k}{4}\|u(t)\|^2. \end{aligned} \tag{5.17}$$

Combining (5.15) – (5.17), one has

$$\lambda^* \ge \left(\frac{d}{E(0)}\right)^{1/2} > 1. \tag{5.18}$$

From (5.15) and (5.18), we deduce

$$I(u(t)) \ge \frac{k}{2} \ln\left(\frac{d}{E(0)}\right) \|u(t)\|^2.$$

The rest estimate is implied by $E(t) < E(0)$ and (5.16), i.e,

$$E(t) \le d \le \frac{k S_2^2}{4}\left(\lambda^*\right)^2 \|\nabla u(t)\|^2 \equiv C \|\nabla u(t)\|^2,$$

where S_2 is the optimal constant of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega).$ The proof follows from (5.15) and (5.18). □

For proving the decay of the solutions, we define the following auxiliary functional

$$L(t) = E(t) + \frac{\varepsilon}{2} \xi(t) \|u(t)\|^2, \tag{5.19}$$

for ε is a small positive number specified later.

The next lemma tells us that $E(t)$ and $L(t)$ are equivalent functions.

Lemma 5.2. For ε small enough, there exist two positive constants α_1, α_2 such that

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t).$$

Proof. From the fact that $u(t)\in\mathcal{W}$ and

$$E(t)=E(u(t))\geq\frac{1}{2}(1-\ell)\|\nabla u(t)\|^2+\frac{1}{2}I(u(t))+\frac{k}{4}\|u(t)\|^2 \tag{5.20}$$

we have

$$\|\nabla u(t)\|^2\leq\frac{2}{1-\ell}E(t)\quad\text{and}\quad\|u(t)\|^2\leq\frac{4}{k}E(t). \tag{5.21}$$

Since $0<\xi(t)\leq\xi(0),$ the above estimates imply that

$$|L(t)-E(t)|\leq\frac{\varepsilon}{2}|\xi(t)|\|u(t)\|^2\leq\frac{2\varepsilon}{k}\xi(0)E(t),$$

that is

$$\left(1-\frac{2\varepsilon}{k}\xi(0)\right)E(t)\leq L(t)\leq\left(1+\frac{2\varepsilon}{k}\xi(0)\right)E(t).$$

By choosing ε small such that $1-\frac{2\varepsilon}{k}\xi(0)>0,$ we claim the lemma. □

The next lemma allows us to estimate $L'(t).$

Lemma 5.3. Let (G), (K) hold. Then for number a such that $0<a<\sqrt{\frac{2\pi\ell}{k}},$ $\varepsilon>0,$ and $0<\Lambda<1,$ we have

$$\begin{aligned}L'(t)\leq&-\varepsilon\Lambda\xi(t)E(t)-\|u_t\|^2-\frac{1}{2}\left(1-\frac{\varepsilon(1-\ell)}{2\delta}-\varepsilon\Lambda\right)\xi(t)\left(g\circ\nabla u\right)(t)\\&-\varepsilon\xi(t)\left[\left(1-\frac{\Lambda}{2}\right)\left(\ell-\frac{ka^2}{2\pi}\right)-\delta\right]\|\nabla u(t)\|^2\\&-\frac{k\varepsilon}{2}\xi(t)\left[-\frac{\Lambda}{2}+\left(1-\frac{\Lambda}{2}\right)\left(-\ln\left(\frac{4}{k}M\right)+n(1+\ln a)+\frac{2}{k}\right)\right]\|u(t)\|^2.\end{aligned} \tag{5.22}$$

Proof. Simple calculations and using Lemma 2.4, we have

$$\begin{aligned}L'(t)=&E'(t)+\varepsilon\xi'(t)\|u(t)\|^2+\varepsilon\xi(t)\int_\Omega u_t(t)u(t)dx\\\leq&-\|u_t\|^2-\frac{1}{2}g(t)\|\nabla u(t)\|^2+\frac{1}{2}\left(g'\circ\nabla u\right)(t)\\&+\varepsilon\xi(t)\left\{-\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2+\int_\Omega u^2\ln|u|^k dx-\|u(t)\|^2\right\}\\&+\varepsilon\xi(t)\int_\Omega\nabla u(t)\cdot\int_0^t g\left(t-s\right)\left(\nabla u(s)-\nabla u(t)\right)dsdx.\end{aligned} \tag{5.23}$$

By using the Young's inequality and note that $1-\int_0^t g(s)ds\geq1-\int_0^\infty g(s)ds=\ell,$ we have

$$\begin{aligned}
&\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)dsdx \\
&\quad\le \|\nabla u(t)\|.\left\|\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)ds\right\| \\
&\quad\le \delta\|\nabla u(t)\|^2+\frac{1}{4\delta}\int_0^t g(s)ds\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2\,ds \\
&\quad\le \delta\|\nabla u(t)\|^2+\frac{1-\ell}{4\delta}\big(g\circ\nabla u\big)(t),
\end{aligned}\tag{5.24}$$

for $\delta>0$.

By using assumption (G, (ii)), (5.24), it implies from (5.23) as follows

$$\begin{aligned}
L'(t)\le -\|u_t\|^2-\left(\frac{1}{2}-\frac{\varepsilon(1-\ell)}{4\delta}\right)\xi(t)\big(g\circ\nabla u\big)(t) \\
-\varepsilon\frac{\Lambda}{2}\xi(t)\hat{I}(t)-\varepsilon\left(1-\frac{\Lambda}{2}\right)\xi(t)\hat{I}(t)+\varepsilon\delta\xi(t)\|\nabla u(t)\|^2,
\end{aligned}\tag{5.25}$$

where

$$\hat{I}(t)=\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2-\int_\Omega u^2\ln|u|^k\,dx+\|u(t)\|^2\ge I(t)=I(u(t)),$$

and $0<\Lambda<1$.

By the fact that

$$\|u(t)\|^2\le\frac{4}{k}M=\left(\frac{2\pi\ell}{k}\right)^{n/2}e^n$$

and using (3.3), we have that

$$\begin{aligned}
I(u)&\ge\left(\ell-\frac{ka^2}{2\pi}\right)\|\nabla u\|^2+\|u\|^2+\frac{kn(1+\ln a)}{2}\|u\|^2-k\|u\|^2\ln\|u\| \\
&\ge\left(\ell-\frac{ka^2}{2\pi}\right)\|\nabla u\|^2+\|u\|^2+\frac{kn(1+\ln a)}{2}\|u\|^2-\frac{k}{2}\|u\|^2\ln\left(\frac{4}{k}M\right).
\end{aligned}\tag{5.26}$$

Furthermore, we imply from (2.3) that

$$\hat{I}(t)=2E(t)-\big(g\circ\nabla u\big)(t)-\frac{k}{2}\|u(t)\|^2.\tag{5.27}$$

By (5.26) and (5.27), it implies from (5.25) that

$$\begin{aligned}
L'(t)\le -\varepsilon\Lambda\xi(t)E(t)-\|u_t\|^2-\frac{1}{2}\left(1-\frac{\varepsilon(1-\ell)}{2\delta}-\varepsilon\Lambda\right)\xi(t)\big(g\circ\nabla u\big)(t) \\
-\varepsilon\xi(t)\left[\left(1-\frac{\Lambda}{2}\right)\left(\ell-\frac{ka^2}{2\pi}\right)-\delta\right]\|\nabla u(t)\|^2 \\
-\frac{k\varepsilon}{2}\xi(t)\left[-\frac{\Lambda}{2}+\left(1-\frac{\Lambda}{2}\right)\left(-\ln\left(\frac{4}{k}M\right)+n(1+\ln a)+\frac{2}{k}\right)\right]\|u(t)\|^2.
\end{aligned}\tag{5.28}$$

The proof is complete. □

Theorem 5.4. Under the assumptions of Theorem 4.2 and g satisfies (G). Then solution $u(t)$ to (1.1) decays exponentially.

Proof. Consider the function

$$\phi(t)=\int_0^t \xi(s)ds,\ t\geq 0.$$

It is clear that ϕ is a non-decreasing function of class C^1 on $\mathbb{R}_+$ and

$$\phi(t)\to+\infty \text{ as } t\to+\infty.$$

First, we choose

$$\max\left\{e^{-1-\frac{1}{n}-\frac{2}{nk}},\ e^{\frac{-2}{nk}}\sqrt{\frac{2\pi\ell}{k}}\right\}<a<\sqrt{\frac{2\pi\ell}{k}}$$

to obtain

$$\ell-\frac{ka^2}{2\pi}>0$$

and

$$-\ln\left(\frac{4}{k}M\right)+n(1+\ln a)+\frac{2}{k}=-\ln\left[\left(\frac{2\pi\ell}{k}\right)^{n/2}e^n\right]+n(1+\ln a)+\frac{2}{k}>0.$$

Next, we choose $\delta>0$ such that

$$\left(\ell-\frac{ka^2}{2\pi}\right)-\delta>0$$

and then choose ε small enough such that

$$1-\varepsilon\frac{1-\ell}{2\delta}>0$$

after that we choose $0<\Lambda<1$ such that

$$1-\frac{\varepsilon(1-\ell)}{2\delta}-\varepsilon\Lambda>0,\ \left(1-\frac{\Lambda}{2}\right)\left(\ell-\frac{ka^2}{2\pi}\right)-\delta>0,$$

and

$$-\frac{\Lambda}{2}+\left(1-\frac{\Lambda}{2}\right)\left(-\ln\left(\frac{4}{k}M\right)+n(1+\ln a)+\frac{2}{k}\right)>0.$$

From Lemma 5.1, 5.3 and the definition of $E(t)$, we can find a positive constant χ such that

$$L'(t)\leq-\chi\xi(t)E(t),\ \forall t\geq 0$$

and thanking Lemma 2.4, we have

$$L'(t)\leq-\frac{\chi}{\alpha_2}\xi(t)L(t),\ \forall t\geq 0$$

which implies

$$L(t)\leq L(0)e^{-\frac{\chi}{\alpha_2}\phi(t)},\ \forall t\geq 0.$$

This completes the proof. □

6. CONCLUSION

The paper is dedicated to study a nonlinear viscoelastic heat equation with logarithmic sources. By introducing a family of potential wells, we prove the global existence and exponential decay for solutions with initial data in the potential wells.

Here the blow up of local solutions of the problem is open, although we prove the non blow up at finite time of solution.

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TÓM TẮT

SỰ TỒN TẠI TOÀN CỤC VÀ TÍNH TẮT DẦN MŨ CỦA NGHIỆM
CỦA MỘT PHƯƠNG TRÌNH NHIỆT ĐÀN HỒI NHỚT VỚI NGUỒN LOGARIT

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Trong bài báo này, nhóm tác giả nghiên cứu một phương trình nhiệt đàn hồi phi tuyến với nguồn logarit. Bằng cách giới thiệu một họ thế vị tốt (potential wells), chúng tôi chứng minh sự tồn tại nghiệm toàn cục và tắt dần mũ của nghiệm với dữ liệu ban đầu trong tập ổn định.

Từ khóa: Phương trình nhiệt phi tuyến đàn hồi nhớt, tồn tại toàn cục, tắt dần mũ, nguồn Lôgarit.