A NOTE ON THE PROPER CONFLICT-FREE CONNECTION NUMBER OF CONNECTED GRAPHS

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Abstract: *Let G be an edge-colored graph. A path in G is a conflict-free path if it contains a color used on exactly one of its edges. The graph G is called conflict-free connected if every two distinct vertices is connected by at least one conflict-free path. The graph G is said to be properly edge colored (properly colored for simplifying), meaning an assignment of colors to edges so that no vertex is incident to two edges of the same color. If graph G is simultaneously properly colored and conflict-free connected, then Czap et al. [1] introduced the concept of properly conflict-free connected. The proper conflict-free connection number, denoted by $pcfc(G)$, is the minimum number of colors needed in order to make it properly conflict-free connected. Recently, there is few results of proper conflict-free connection. In this paper, we determine some connected graph classes having $pcfc(G) = \chi'(G)$.*

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I. INTRODUCTION

In this paper, we only consider simple, finite, and undirected graphs. Given a connected graph G. Let us denote by $V(G), E(G), n, m$ the vertex set, the edge set, the number of vertices, the number of edges of G, respectively. Let $u, v \in V(G)$ be two vertices of G and $N_G(u)$ be the neighbor set of the vertex u in G. A path connecting two vertices u, v is denoted by uv-path. The length of the uv-path is its number of edges. The distance between two vertices u, v, denoted by $d(u, v)$, is the length of the shortest path between u, v. The diameter of G, denoted by $diam(G)$, is the maximum distance between any pair vertices of G. Finally, we use [2] for some terminology and notation not defined here.

Let k be an integer. For simplifying notation, we denote the set $\{1,2,\dots,k\}$ by $[k]$. Let $c: E(G) \rightarrow [k]$ be an edge-colored of G i.e., every edge of G is assigned by a color from $[k]$.

Given an edge-colored graph G. If adjacent edges in G are assigned different colors by c then G is called *properly edge-colored* (properly colored for simplifying). Now, c is the *proper edge-coloring*. The *chromatic index number*, denoted by $\chi'(G)$, is the least number of colors needed in order to make it properly colored. The bounds of the chromatic index number were determined by Vizing et al. [3] as follows: $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Moreover, if $\chi'(G) = \Delta(G)$ then G is in *Class 1*. Otherwise, G is in *Class 2*. Holyer et al. [4] presented that it is NP-complete to determine the chromatic index of an arbitrary graph.

Connectivity is one of the most fundamental in graph theory, both in the combinatorial sense and the algorithmic sense. There exist many beautiful and interesting results on connectivity in graph theory. The first connection concept is *rainbow connection* that was introduced by Chartrand et al. [5]. This concept has many applications in communication and secure of networks. Readers who are interested in this topic are referred to [6,7] .

In 2018, Czap et al. [1] introduced the concept of *conflict-free connection*. Let G be a nontrivial, connected, edge-colored graph. The edge-colored graph G is said to be *conflict-free connected* if any two vertices are connected by a path which contains at least one color used on exactly one of its edges. This path is called *conflict-free path*. The *conflict-free connection number* of G, denoted by $cfc(G)$, is defined as the smallest number of colors in order to make it conflict-free connected. For more results on conflict-free connection, readers are referred to [1,8-10]. Very recently, it has been shown in [11] that computing $cfc(G)$ for a given graph G is an NP-hard problem.

The last section in [1], the authors introduced the concept of *proper conflict-free connection*. If G is simultaneously properly colored and conflict-free connected, then G is called *properly conflict-free connected*. The *proper conflict-free connection number*, denoted by $pcfc(G)$, is the minimum number of colors in order to make G properly connected and conflict-free connected. Clearly, $pcfc(G) \geq \max\{\chi'(G), cfc(G)\}$.

The concept of proper conflict-free connection is very new. Recently, there are only some results on the proper conflict-free connection number in [1]. Therefore, in this paper, we consider the proper conflict-free connection number of some special connected graphs. Moreover, we also determine some connected graph classes having $pcfc(G) = \chi'(G)$.

2. CONTENT

2.1. Auxiliary results

In this section, we present some known results which are very useful in the proof of our results. First, we state results on the conflict-free connection number.

Theorem 2.1. ([1]). *Let G be a path with $p = n - 1 \geq 1$ edges. Then $cfc(G) = \lceil log_2(p + 1) \rceil$.*

Czap et al. [1] determined the conflict-free connection number of 2-connected graphs as follows.

Lemma 2.2. ([1]). *If G is a 2-connected and non-complete graph, then $cfc(G) = 2$.*

Next, we present some results on the proper conflict-free connection number. In [1], the authors said that, for any tree T, there is $pcfc(T) = cfc(T)$. For any 2-connected graph, the bounds of the proper conflict-free connection number are determined as follows.

Theorem 2.3. ([1]). *If G is a 2-connected graph, then*

$$\Delta(G) \leq \chi'(G) \leq pcfc(G) \leq \chi'(G) + 1 \leq \Delta(G) + 2.$$

For an arbitrary connected graph, the upper bound of the proper conflict-free connection number is determined.

Theorem 2.4. ([1]). *If G is a connected graph with $\Delta^*(G) = \Delta(G - E(C(G)))$ and $h(G) = max\{cfc(K): K \text{ is a component of } C(G)\}$ then*

$$pcfc(G) \leq \Delta^*(G) + h(G) + 2.$$

Finally, we present some results on the chromatic index number.

Theorem 2.5. ([12]). *If G is a bipartite graph, then G is in Class 1.*

Theorem 2.6. ([13]). *The complete graph K_n is in Class 1 if n is even and in Class 2 if n is odd.*

2.2. Main results

In this section, we present some new results of the proper conflict-free connection number. Given two integers n, m. Let K_n, where $n \geq 2$, $K_{m,n}$, where $m \geq n \geq 1$, be a complete graph, a complete bipartite graph, respectively. First, we present a class of connected graphs, say G, with the condition of diameter having $pcfc(G) = \chi'(G)$.

Proposition 3.1. *Let G be a connected graph of order $n \geq 2$. If $diam(G) \leq 3$ then $pcfc(G) = \chi'(G)$.*

Proof. Let $c: E(G) \rightarrow \chi'(G)$ be a proper edge-colored of G. Now we show that every two distinct vertices, say $u, v \in V(G)$, is connected by at least one conflict-free path. If $uv \in E(G)$ then uv is a conflict-free path connecting u, v. Hence, we consider $uv \notin E(G)$. It follows that $2 \leq d(u, v) \leq 3$.

1. If $d(u, v) = 2$ then there exists a vertex w such that $w \in N_G(u) \cap N_G(v)$. Since G is properly colored, $c(wu) \neq c(wv)$. It implies that uwv is a conflict-free path.
2. If $d(u, v) = 3$ then there exists a path of length 3, say $P = uw_1w_2v$, connecting u, v. Since G is properly colored, $c(w_1u) \neq c(w_1w_2)$ and $c(w_2w_1) \neq c(w_2v)$. It can be readily seen that P is a conflict-free path.

Hence, there always exists at least one conflict-free path connecting any two distinct vertices in the properly edge-colored graph G. By the definition of proper conflict-free connection, it follows $pcfc(G) \leq \chi'(G)$. On the other hand, $pcfc(G) \geq max\{\chi'(G), \Delta(G)\} \geq \chi'(G)$. Hence, $pcfc(G) = \chi'(G)$.

The proof is completed.

Sharpness: The bound of $diam(G)$ of Proposition 3.1 is best possible. Let $G_1 \cong P_5$ be a path of order 5. Clearly, $diam(G_1) = 4$ and $\chi'(G_1) = 2$. By Theorem 2.1, $cfc(G_1) = 3$. Hence, $pcfc(G_1) \geq max\{2,3\} = 3$.

Next, we consider the proper conflict-free connection number of a complete graph and a complete bipartite graph.

Theorem 3.1. *Let G be a connected graph. Hence,*

1. *if $G \cong K_n$, where $n \geq 2$ then*

$$pcfc(K_n) = \begin{cases} n-1 & if\ n\ is\ even; \\ n & if\ n\ is\ odd \end{cases};$$

2. *if $G \cong K_{m,n}$, where $m, n \geq 1$, then $pcfc(K_{m,n}) = max\{m, n\}$;*

Proof. Clearly, $diam(G) \leq 2$, where $G \in \{K_n, K_{m,n}\}$. By Proposition 3.1, $pcfc(G) = \chi'(G)$.

1. if $G \cong K_n$, where $n \geq 2$. Clearly, $\Delta(G) = n - 1$. By Theorem 2.6, we obtain the result.
2. if $G \cong K_{m,n}$, where $m, n \geq 1$. $\Delta(G) = max\{m, n\}$. By Theorem 2.5, we obtain the result.

The proof is completed.

Let C_n be a cycle of order $n \geq 3$. By Theorem 3.1, $pcfc(C_3) = 3$. Next, the proper conflict-free number of a cycle is presented as follows.

Theorem 3.3. *If C_n is a cycle of order $n \geq 4$, then*

1. *$pcfc(C_n) = 2$ if $n \in \{4,6\}$;*
2. *$pcfc(C_n) = 3$ if $n \geq 5$ and $n \neq 6$.*

Proof. It is not difficult to show that $pcfc(C_n) = 2$, where $n \in \{4,6\}$.

Next, we consider $pcfc(C_n)$, where $n \geq 5$ and $n \neq 6$. Let us assign all edges of C_n by colors from [3] as follows: $c(v_i v_{i+1}) = ((i-1) \mod 2) + 1, \forall i \in [n-1]$; $c[v_n v_1] = 3$. It can be readily seen that C_n is simultaneously properly colored and conflict-free connected. Hence, $pcfc(C_n) \leq 3$.

If n is odd, then $\chi'(C_n) = 3$. Hence, $pcfc(C_n) \geq 3$. It follows $pcfc(C_n) = 3$.

If n is even, then $n = 2k$, where $k \geq 4$ and $\chi'(C_{2k}) = 2$. Hence, $pcfc(C_{2k}) \geq 2$. Suppose that $pcfc(C_{2k}) = 2$. Let $C_{2k} = v_1 \ldots v_{2k} v_1$. Since C_{2k} is properly conflict-free connected with two colors from [2], two consecutive edges of C_{2k} must be assigned different colors. On the other hand, $d(v_1 v_{k+1}) \geq 4$. By Lemma 2.1, $v_1 v_{k+1}$-path is not able to be conflict-free path with two colors, a contradiction. Hence, $pcfc(C_{2k}) \geq 3$. This implies that $pcfc(C_{2k}) = 3$.

Our proof is obtained.

Let W_n be a wheel of order $n+1$ consisting of a cycle C_n and a single vertex u joined to all vertices of the cycle C_n. The following result is shown that different from a cycle C_n, the proper conflict-free connection number of W_n is as large as possible.

Theorem 3.4. *Let* $n \geq 4$ *be an integer. Hence,* $pcfc(W_n) = n$.

Proof. Let $W_n = C_n + u$ be a wheel, where $C_n = v_1 \dots v_n v_1$. Let c be an edge-coloring of W_n. Clearly, $\Delta(W_n) = n$. Let us assign all edges of W_n by n colors as follows: $c(uv_i) = i, \forall i \in [n]$; $c(v_i v_{i+1}) = ((i+1)\mathrm{mod} n) + 1, \forall i \in [n]$ (indices modulo n). It can be readily seen that W_n is properly colored. Hence, $\chi'(W_n) \leq n$. On the other hand, $\chi'(W_n) \geq \Delta(W_n)$. It follows that $\chi'(W_n) = \Delta(W_n) = n$.

Clearly, $diam(W_n) = 2$, using Proposition 3.1, $pcfc(W_n) = \chi'(G) = n$.

The proof is completed.

Let $\overline{G}$ be the complement of G. Now, we consider the conflict-free connection number of the complement of G with the condition of its diameter.

Theorem 3.5. *If* G *is a graph of order* $n \geq 3$, *where* $diam(G) \geq 3$, *then* $pcfc(\overline{G}) = \chi'(\overline{G})$.

Proof. Since $diam(G) \geq 3$, there exist two vertices, say $u, v \in V(G)$, such that $uv \notin E(G)$. This makes $uv \in E(\overline{G})$. Moreover, u, v have no common neighbors in G. Hence, every $w \in V(G) \backslash \{u, v\}$ has at least one vertex from $\{u, v\}$ as non-neighbor in G. This deduces that w is adjacent to at least one vertex from $\{u, v\}$ in $\overline{G}$. It can be readily seen that there is at least one path in $\overline{G}$ connecting any two arbitrary vertices $x, y \in V(\overline{G})$. Hence, $\overline{G}$ is connected. There exists $pcfc(\overline{G})$.

Moreover, $d_{\overline{G}}(x, y) \leq 3$, for any $x, y \in V(\overline{G})$. It follows that $diam(\overline{G}) \leq 3$. By Proposition 3.1, $pcfc(\overline{G}) = \chi'(\overline{G})$.

This completes our proof.

We finish this section here.

3. CONCLUSION

The concept of the proper conflict-free connection number, say $pcfc(G)$, of a connected graph G is very new. This has been introduced very recently by Czap et. al. in [1]. In this paper, we proved some new results on this concept such as: $pcfc(G) = \chi'(G)$ when $diam\ (G) \leq 3$, $pcfc(G)$ of some special classes of connected graphs… Since $pcfc(G) \geq max\{\chi'(G), cfc(G)\}$ and Lemma 2.1, we deduce that there are many connected graphs, say G, for example $G \cong P_n$ of order $n \geq 5$, such that $pcfc(G) = cfc(G) > \chi'(G)$.

Let C_n be an even cycle of order $n \geq 8$. Hence, $\chi'(C_n) = 2$ and $cfc(C_n) = 2$ (by Lemma 2.2). By Theorem 3.3, $pcfc(C_n) = 3 > max\{\chi'(C_n), cfc(C_n)\}$. Hence, there exists a connected graph G such that $pcfc(G) \geq max\{\chi'(G), cfc(G)\} + 1$. It is natural the following problem is raised.

Problem 3.6. *Let $t \geq 2$. Is there any nontrivial, connected graph G having*

$$pcfc(G) \geq max\{\chi'(G), cfc(G)\} + t.$$

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MỘT LƯU Ý VỀ SỐ KẾT NỐI KHÔNG XUNG ĐỘT THÍCH HỢP CỦA CÁC ĐỒ THỊ LIÊN THÔNG

Tóm tắt: *Cho $\boldsymbol{G}$ là một đồ thị được tô màu tất cả các cạnh. Một đường trong đồ thị $\boldsymbol{G}$ được gọi là đường không xung đột nếu trên đó có một màu sử dụng chính xác một lần duy nhất. Đồ thị $\boldsymbol{G}$ được gọi là liên thông không xung đột nếu cứ hai đỉnh khác nhau bất kỳ trong đồ thị được kết nối với ít nhất một đường không xung đột. Đồ thị $\boldsymbol{G}$ được gọi là tô màu thích hợp, khi đó các màu được tô cho các cạnh của đồ thị sao cho không có cạnh nào cùng kề với một đỉnh nhận màu giống nhau. Nếu đồ thị $\boldsymbol{G}$ vừa là đồ thị thỏa mãn đồng thời điều kiện tô màu thích hợp và liên thông không xung đột thì các tác giả tại [1] đã giới thiệu khái niệm mới là liên thông không xung đột thích hợp. Số liên thông không xung đột thích hợp, được ký hiệu là $\boldsymbol{pcfc}(\boldsymbol{G})$, là số màu nhỏ nhất cần phải tô tất cả các cạnh của đồ thị sao cho đồ thị trở thành liên thông không xung đột thích hợp. Hiện nay, có rất ít kết quả về số liên thông không xung đột thích hợp. Trong bài báo này, chúng tôi xác định số liên thông không xung đột thích hợp của một số lớp đồ thị liên thông thỏa mãn $\boldsymbol{pcfc}(\boldsymbol{G}) = \boldsymbol{\chi}'(\boldsymbol{G})$.*

Từ khoá: *Liên thông không xung đột, màu sắc thích hợp.*