TẠP CHÍ KHOA HỌC & CÔNG NGHỆ ĐẠI HỌC DUY TÂN
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Overview of Legendre-Fenchel duality

Tổng quan về đối ngẫu Legendre-Fenchel

Tina Mai [a,b,*]
Mai Ti Na [a b *]

a. Institute of Research and Development, Duy Tan University, Da Nang, 550000, Vietnam
Viện Nghiên cứu và Phát triển Công nghệ cao, Đại học Duy Tân, Đà Nẵng, 550000, Việt Nam
b. Faculty of Natural Sciences, Duy Tan University, Da Nang, 550000, Vietnam
Khoa Khoa học Tự nhiên, Đại học Duy Tân, Đà Nẵng, 550000, Việt Nam

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Abstract

We give some overview of Legendre-Fenchel duality.

Keywords: Legendre-Fenchel duality.

Tóm tắt

Chúng tôi đưa ra một vài tổng quan về đối ngẫu Legendre-Fenchel.

Từ khóa: Đối ngẫu Legendre-Fenchel.

1. Introduction

Legendre-Fenchel duality plays a helpful role in convex optimization. Herein, we introduce some overview of Legendre-Fenchel duality, with an eye toward later applications in nonlinear elasticity. The basic tool here is functional analysis.

2. Preliminaries

In this paper, we work with real field. The notations here are as introduced in [1]. The dual space of normed vector space X is denoted by X^*, with the associated duality ${}_{X^*}\langle \cdot, \cdot \rangle_X$. The bidual space of X is denoted by X^{**}. In case X is a reflexive Banach space, X^{**} will coincide with X by means of the usual canonical isometry.

Let A be a subset of X. The *indicator function* of A is defined by

$$I_A(x) := \begin{cases} 0 \text{ if } x \in A, \\ +\infty \text{ if } x \notin A. \end{cases}$$

A function $g : X \to \mathbb{R} \cup \{+\infty\}$ is *proper* if $\{x \in X | g(x) < +\infty\} \neq \emptyset$.

Let $\mathbf{\Sigma}$ be a normed vector space and let $g : \mathbf{\Sigma} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The

* *Corresponding Author: Tina Mai; Institute of Research and Development, Duy Tan University, Da Nang, 550000, Vietnam; Faculty of Natural Sciences, Duy Tan University, Da Nang, 550000, Vietnam.*
Email: maitina@duytan.edu.vn

Legendre-Fenchel transform of g is the function

$$g^* : \boldsymbol{\Sigma}^* \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$g^* : \boldsymbol{\epsilon} \in \boldsymbol{\Sigma}^* \to g^*(\boldsymbol{\epsilon}) := \sup_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} \{ {}_{\boldsymbol{\Sigma}^*}\langle \boldsymbol{\epsilon}, \boldsymbol{\sigma} \rangle_{\boldsymbol{\Sigma}} - g(\boldsymbol{\sigma}) \} .$$

In nonlinear elasticity, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ represent the traditional stress and strain, respectively.

3. Legendre-Fenchel duality

We consider a given reflexive Banach space $\boldsymbol{\Sigma}$. The next theorem summaries some basic properties of the Legendre-Fenchel transform. We refer the readers to [1, 2] for the statement and proof.

Theorem 3.1 ([1, 2])**.** *Let $\boldsymbol{\Sigma}$ be a reflexive Banach space, and given $g : \boldsymbol{\Sigma} \to \mathbb{R} \cup \{+\infty\}$ a proper, strictly convex, and lower semi-continuous function. Then, the Legendre-Fenchel transform $g^* : \boldsymbol{\Sigma}^* \to \mathbb{R} \cup \{+\infty\}$ of g is also proper, strictly convex, and lower semi-continuous. Let*

$$g^{**} : \boldsymbol{\sigma} \in \boldsymbol{\Sigma}^{**} \to g^{**}(\boldsymbol{\sigma}) := \sup_{\boldsymbol{\epsilon} \in \boldsymbol{\Sigma}^*} \{ {}_{\boldsymbol{\Sigma}^*}\langle \boldsymbol{\epsilon}, \boldsymbol{\sigma} \rangle_{\boldsymbol{\Sigma}} - g^*(\boldsymbol{\epsilon}) \}$$

denote the Legendre-Fenchel transform of g^*. Then, (with $X^{**} \equiv X$),

$$g^{**} = g .$$

The equality $g^{**} = g$ forms the *Fenchel-Moreau theorem.*

Given a *minimization problem* ($\mathscr{P}$) with

$$\inf_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} G(\boldsymbol{\sigma}) , \tag{1}$$

provided a function $G : \boldsymbol{\Sigma} \to \mathbb{R} \cup \{+\infty\}$ of the specific form given in Theorem 3.2, the following result will be the basis for defining *two different dual problems of problem ($\mathscr{P}$) with (1).* The proof is based on Theorem 3.1 and can be found in [1].

Theorem 3.2 ([1])**.** *Let $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ be two reflexive Banach spaces, and given $g : \boldsymbol{\Sigma} \to \mathbb{R} \cup \{+\infty\}$ and $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ two proper, strictly convex, and lower semi-continuous functions, let $\Lambda : \boldsymbol{\Sigma} \to \boldsymbol{V}^*$ be a linear and continuous mapping. Let the function $G : \boldsymbol{\Sigma} \to \mathbb{R} \cup \{+\infty\}$ be defined by.*

$$G : \boldsymbol{\sigma} \in \boldsymbol{\Sigma} \to G(\boldsymbol{\sigma}) := g(\boldsymbol{\sigma}) + h(\Lambda \boldsymbol{\sigma}) .$$

Finally, let the two Lagrangians associated with the minimization problem ($\mathscr{P}$).

$$\mathscr{L} : \boldsymbol{\Sigma} \times \boldsymbol{\Sigma}^* \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

and

$$\tilde{\mathscr{L}} : \boldsymbol{\Sigma} \times \boldsymbol{V} \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

be defined by

$$\mathscr{L} : (\boldsymbol{\sigma}, \boldsymbol{\epsilon}) \in \boldsymbol{\Sigma} \times \boldsymbol{\Sigma}^* \to \mathscr{L}(\boldsymbol{\sigma}, \boldsymbol{\epsilon})$$

where

$$\mathscr{L}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}) := {}_{\boldsymbol{\Sigma}^*}\langle \boldsymbol{\epsilon}, \boldsymbol{\sigma} \rangle_{\boldsymbol{\Sigma}} - g^*(\boldsymbol{\epsilon}) + h(\Lambda \boldsymbol{\sigma}) ,$$

and

$$\tilde{\mathscr{L}} : (\boldsymbol{\sigma}, \boldsymbol{v}) \in \boldsymbol{\Sigma} \times \boldsymbol{V} \to \tilde{\mathscr{L}}(\boldsymbol{\sigma}, \boldsymbol{v})$$

where

$$\tilde{\mathscr{L}}(\boldsymbol{\sigma}, \boldsymbol{v}) := g(\boldsymbol{\sigma}) + {}_{\boldsymbol{V}^*}\langle \Lambda \boldsymbol{\sigma}, \boldsymbol{v} \rangle_{\boldsymbol{V}} - h^*(\boldsymbol{v}) .$$

Then,

$$\inf_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} G(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} \sup_{\boldsymbol{\epsilon} \in \boldsymbol{\Sigma}^*} \mathscr{L}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}) = \inf_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} \sup_{\boldsymbol{v} \in \boldsymbol{V}} \tilde{\mathscr{L}}(\boldsymbol{\sigma}, \boldsymbol{v}) .$$

In our case, as in [1], the dual problem corresponding to the first inf-sup problem found in Theorem 3.2 is defined as problem ($\mathscr{P}^*$) with

$$\sup_{\boldsymbol{\epsilon} \in \boldsymbol{\Sigma}^*} G^*(\boldsymbol{\epsilon}) ,$$

where

$$G^*(\boldsymbol{\epsilon}) := \inf_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} \{ {}_{\boldsymbol{\Sigma}^*}\langle \boldsymbol{\epsilon}, \boldsymbol{\sigma} \rangle_{\boldsymbol{\Sigma}} + h(\Lambda \boldsymbol{\sigma}) \} - g^*(\boldsymbol{\epsilon}) \quad \forall \boldsymbol{\epsilon} \in \boldsymbol{\Sigma}^* . \tag{2}$$

The dual problem corresponding to the second sup-inf problem is defined as problem ($\tilde{\mathscr{P}}^*$) with

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}} \tilde{G}^*(\boldsymbol{v}) ,$$

where

$$\tilde{G}^*(\boldsymbol{v}) := \inf_{\boldsymbol{\sigma}\in\boldsymbol{\Sigma}} \{g(\boldsymbol{\sigma}) + {}_{\boldsymbol{\Sigma}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}}\} - h^*(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{3}$$

A key matter then includes deciding whether the infimum found in problem ($\mathscr{P}$) with (1) is equal to the supremum found in either one of its dual problems.

If this is the case, the next issue consists of identifying whether the Lagrangian $\mathscr{L}$ has a *saddle-point* $(\boldsymbol{T}, \boldsymbol{E}) \in \boldsymbol{\Sigma} \times \boldsymbol{\Sigma}^*$.

4. Conclusions

In this paper, we introduce some overview of Legendre-Fenchel duality, in the spirit of convex optimization. We wish to later apply this knowledge to nonlinear elasticity in three-dimensional settings. The main tool here is functional analysis.

References

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